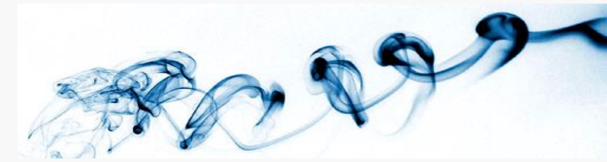
ME 321: Fluid Mechanics-I

Prof. Dr. A.B.M. Toufique Hasan
Department of Mechanical Engineering
Bangladesh University of Engineering and Technology (BUET)

Lecture - 06 (24/05/2025)
Fluid Dynamics: Linear Momentum Equation

toufiquehasan.buet.ac.bd
toufiquehasan@me.buet.ac.bd



Recap

Reynolds transport theorem (RTT) for a fixed, nondeforming control volume (CV)

$$\boxed{\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\, \rho\, d\mathcal{V}\right) + \int_{\text{CS}} \beta\, \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA}$$

This relation permits to change from a system approach to control volume (CV) approach.

where

$$B_{\text{syst}} = \text{any property of fluid (mass, momentum, enthalpy, etc.)}$$

$$\beta = \text{intensive property of fluid (per unit mass basis)}$$

$$\rho = \text{density of fluid}$$

$$d\mathcal{V} = \text{elemental volume}$$

$$\left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA = \text{elemental volume flux}$$

$$\int_{\text{CV}} = \text{volume integral over the control volume (CV)}$$

$$\int_{\text{CS}} = \text{surface integral over the control surface (CS)}$$

Similar expression adopted by other books:

$$\boxed{\frac{D}{Dt}\left(B_{\text{syst}}\right) = \frac{\partial}{\partial t}\left(\int_{\text{CV}} \beta\, \rho\, d\mathcal{V}\right) + \int_{\text{CS}} \beta\, \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA}$$

Conservation of linear momentum

Consider, momentum as a system property

$$B = m\vec{\mathbf{V}} \qquad \text{(momentum)} \qquad\qquad \therefore\ \beta = \frac{m\vec{\mathbf{V}}}{m} = \vec{\mathbf{V}}$$

RTT takes the form of

$$\boxed{\frac{d\left(m\vec{\mathbf{V}}\right)_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \vec{\mathbf{V}}\rho\, d\forall + \int_{\text{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\boxed{\vec{\mathbf{V}} = u\hat{i} + v\hat{j} + w\hat{k} = (u, v, w)}$$

Newton’s second law of motion for a system is

Time rate of change of the linear momentum of the system **=** Sum of external forces acting on the system

$$\boxed{\frac{d\left(m\vec{\mathbf{V}}\right)_{\text{sys}}}{dt} = \sum \vec{F}_{\text{sys}} = \sum \vec{F}_{\substack{\text{contents of the}\\ \text{control volume}}}}$$

Since when a control volume coincident with a system at an instant of time, the forces acting on the system and the forces acting on the contents of the coincident control volume are instantaneously identical.

Conservation of linear momentum

For a control volume (CV) which is fixed and nondeforming, the Newton's second law of motion takes the following form:

$$\sum \vec{F}_{\substack{\text{contents of the}\\ \text{control volume}}} = \frac{d}{dt}\int_{\text{CV}} \vec{\mathbf{V}}\rho\, d\forall + \int_{\text{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

Force contents of the control volume (CV) = Time rate of change of the linear momentum of the contents of the control volume (CV) + Net rate of linear momentum through the control surface (CS)

In fluid mechanics, there are two types of forces are to be considered,

(i) surface force, F_S which acts on the surfaces on the CV (pressure and viscous shear stress)

(ii) body force, F_B which acts on the mass content of the CV (weight, electromagnetic force, etc.)

Then,

$$\sum\left(\vec{F}_S + \vec{F}_B\right) = \frac{d}{dt}\int_{\text{CV}} \vec{\mathbf{V}}\rho\, d\forall + \int_{\text{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

**** Vector equation**

This is known as the **momentum equation**
Or **equation of motion** **<u>in integral form</u>**
applicable for fluid dynamics.

Conservation of linear momentum

The momentum equation is a vector equation. Considering 3 components in Cartesian coordinate system (x, y, z), the momentum equation in integral form comes as:

$$x-\text{direction}\ : \sum\left(F_{S_x}+F_{B_x}\right)=\frac{d}{dt}\int_{\text{CV}} u\rho\, d\forall + \int_{\text{CS}} u\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$y-\text{direction}\ : \sum\left(F_{S_y}+F_{B_y}\right)=\frac{d}{dt}\int_{\text{CV}} v\rho\, d\forall + \int_{\text{CS}} v\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$z-\text{direction}\ : \sum\left(F_{S_z}+F_{B_z}\right)=\frac{d}{dt}\int_{\text{CV}} w\rho\, d\forall + \int_{\text{CS}} w\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\vec{\mathbf{V}}=u\hat{i}+v\hat{j}+w\hat{k}=(u,v,w)$$

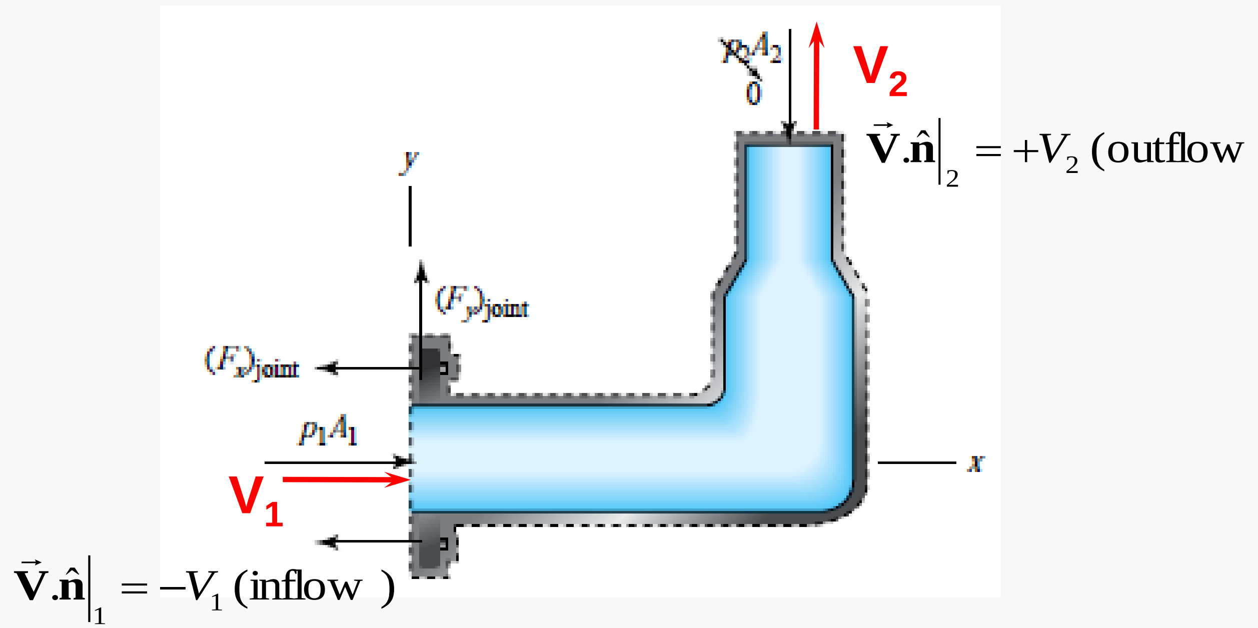
Momentum Principle

The above expression could be simplified considerably if a flow system has **only one entrance and one exit and if the flow is steady**:

$$\sum \vec{F}_{CV} = \cancel{\frac{d}{dt}\int_{CV} \vec{\mathbf{V}}\rho\, d\mathcal{V}} + \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \sum \vec{F}_{CV} = \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \sum \vec{F}_{CV} = \rho_2 A_2 V_2 \vec{\mathbf{V}}_2 - \rho_1 A_1 V_1 \vec{\mathbf{V}}_1$$



Using continuity:

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad \text{(mass flow rate)}$$

Then:

$$\boxed{\sum \vec{F}_{CV} = \dot{m}\left(\vec{\mathbf{V}}_2 - \vec{\mathbf{V}}_1\right)}$$

Note that the momentum equation is a vector equation which represents three scalar equations:

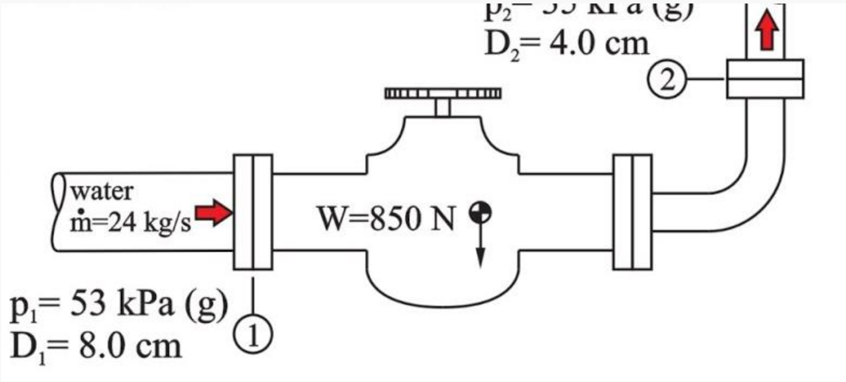
$$x: \quad \sum F_x = \dot{m}\left(V_{2x} - V_{1x}\right)$$
$$y: \quad \sum F_y = \dot{m}\left(V_{2y} - V_{1y}\right)$$
$$z: \quad \sum F_z = \dot{m}\left(V_{2z} - V_{1z}\right)$$

This is the fundamental principle which drives the turbomachinery (propulsion nozzle in jet engine, turbine, compressor cascade etc.)

Applications of Momentum Principle

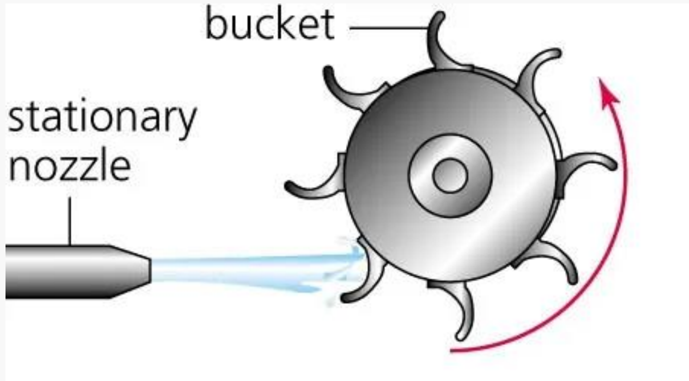
Pipe Bends:

Analyzing the force on a pipe bend involves understanding how the fluid's momentum changes as it navigates the bend, influencing the forces on the pipe itself.



Nozzles and Jet Impact:

The principle is crucial for determining the force a jet of fluid exerts when it impacts a surface, whether it's a plate or a curved vane, like in some types of turbines.

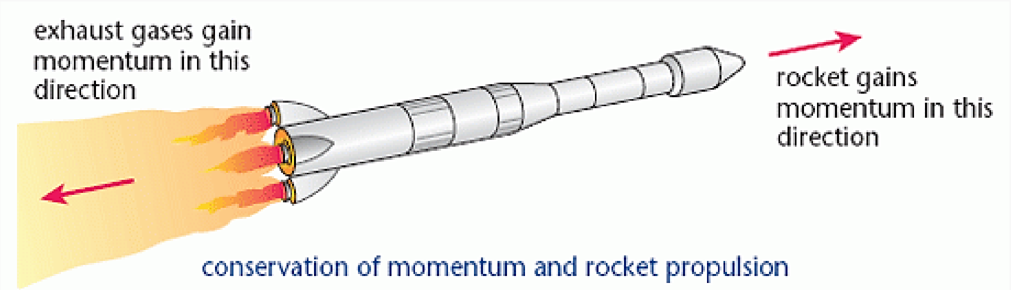


Moving Vanes:

In turbo machines, understanding how a jet of fluid interacts with moving vanes is vital. The momentum principle helps determine the forces acting on the vanes and the work done by the fluid.

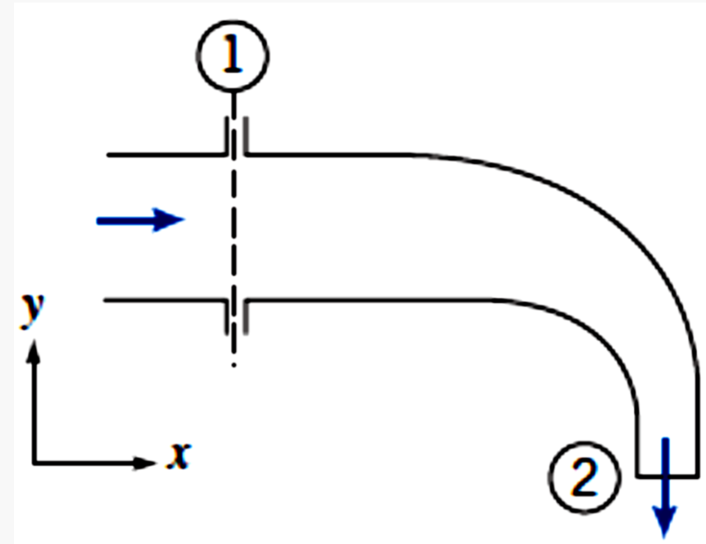
Jet Propulsion:

The same principle applies to jet propulsion, where the change in momentum of the exhaust gases provides thrust.



Problem # 1

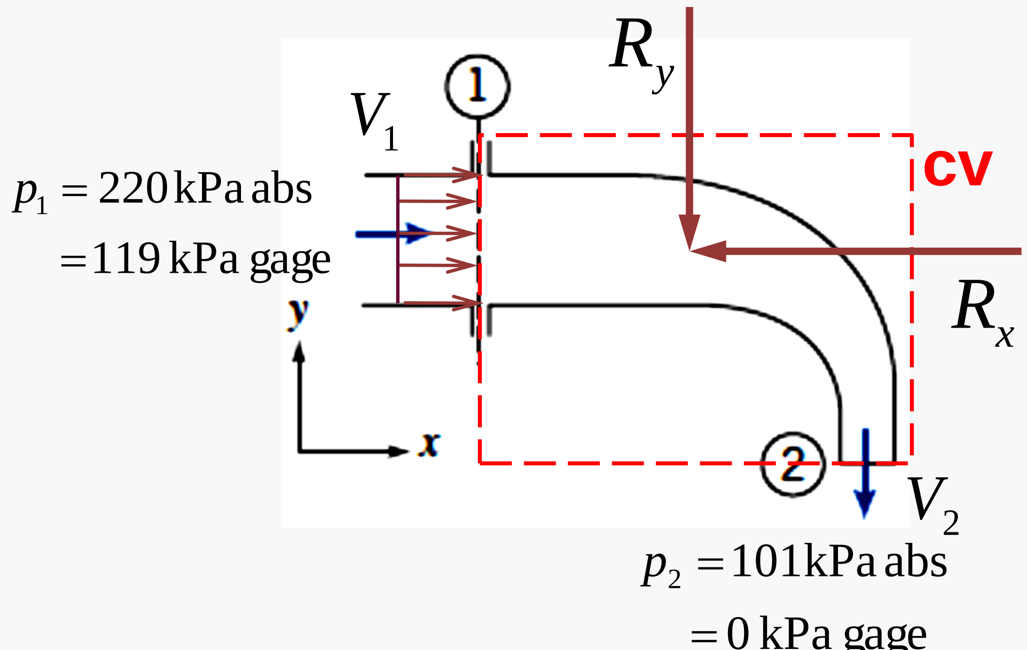
Water flows steadily through the 90° reducing elbow as shown in figure. At the inlet of the elbow, the absolute pressure is 220 kPa and the cross-sectional area is 0.01 m². At the outlet, the cross-sectional area is 0.0025 m² and the velocity is 16 m/s. The elbow discharges to the atmosphere. Determine the force required to hold the elbow in place.



Solution:

From continuity equation:

$$Q_1 = Q_2 \quad \Rightarrow A_1 V_1 = A_2 V_2$$
$$\Rightarrow (0.01) V_1 = (0.0025)(16)$$
$$\therefore V_1 = 4\ \text{m/s}$$



from steady flow momentum principle:

$$\boxed{\sum \vec{F}_{CV} = \int_{CS} \vec{\mathbf{V}} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}} \right) dA}$$

$$\rightarrow x: \sum F_x = \dot{m}\left(V_{2x} - V_{1x}\right)$$
$$\Rightarrow -R_x + p_1 A_1 = \dot{m}\left(V_{2x} - V_{1x}\right) = \dot{m}\left(0 - V_{1x}\right) = -\left(\rho V_1 A_1\right) V_{1x}$$
$$\Rightarrow R_x = \left(\rho V_1 A_1\right) V_1 + p_1 A_1$$

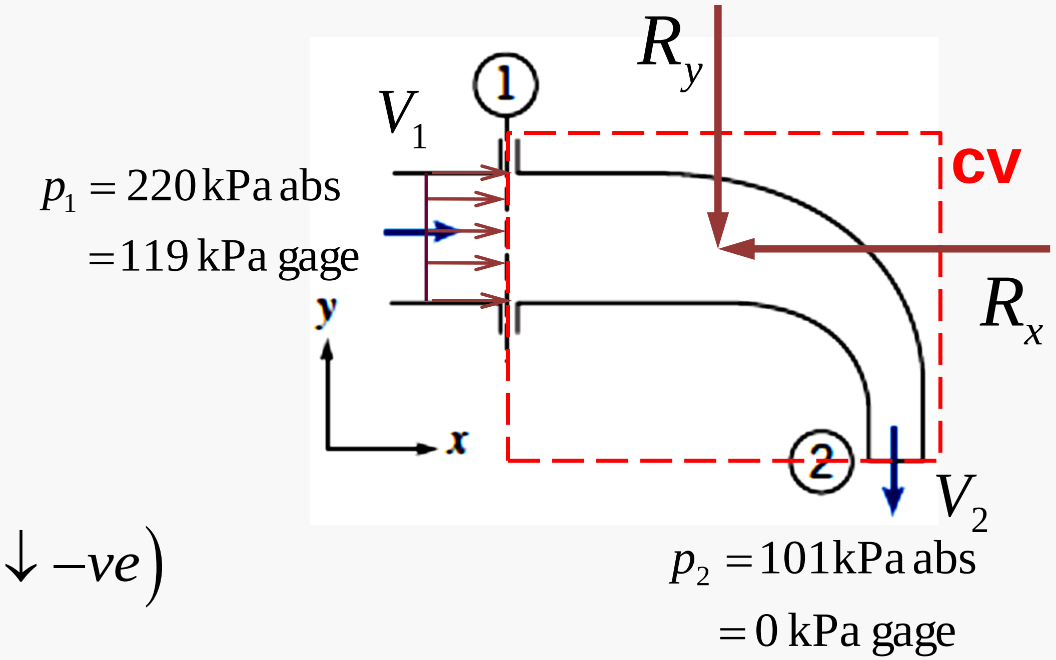
Problem # 1 cont...

Then

$$\Rightarrow R_x = (\rho V_1 A_1)V_1 + p_1 A_1$$

$$\Rightarrow R_x = (119\times10^3)(0.01) + (1000\times4\times0.01)(4)$$

$$\therefore R_x = 1.35\,\text{kN}$$



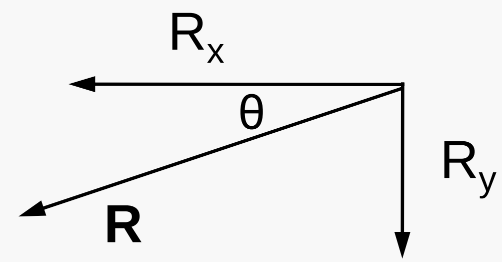
Now, along y-axis:

$$\uparrow y:\ \sum R_y = \dot{m}\left(V_{2y} - V_{1y}\right)$$

$$\Rightarrow -R_y = \dot{m}\left(V_{2y} - V_{1y}\right) = \dot{m}\left(-V_{2y} - 0\right) = -(\rho V_2 A_2)V_2 \quad ;\left(V_2 \downarrow -ve\right)$$

$$\Rightarrow R_y = (1000\times16\times0.0025)(16)$$

$$\therefore R_y = 0.64\,\text{kN}$$



So, the magnitude of resultant force is

$$R = \sqrt{R_x^2 + R_y^2}$$

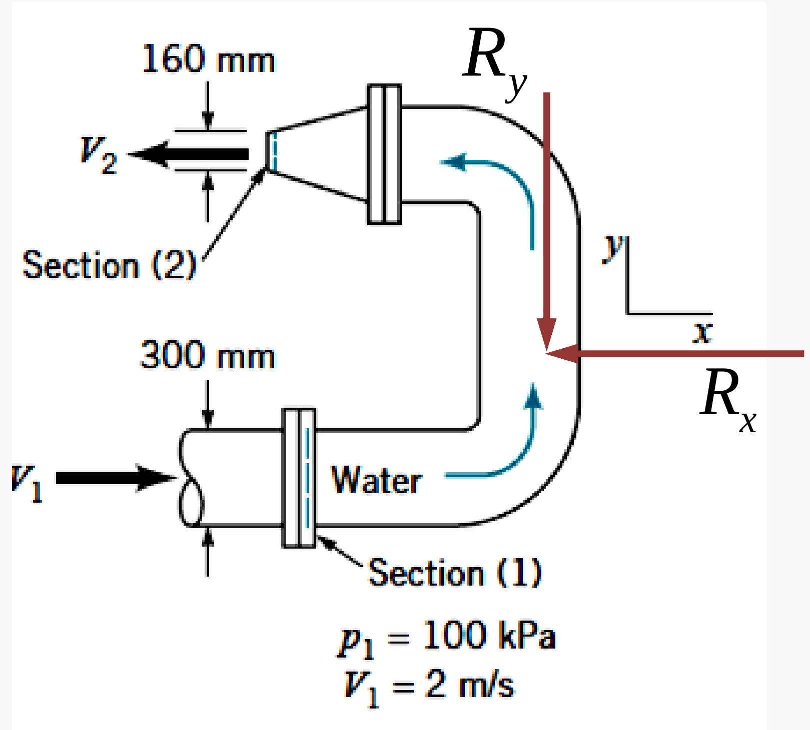
$$\Rightarrow R = \sqrt{1.35^2 + 0.64^2}$$

$$\Rightarrow R = 1.49\,\text{kN}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = 25.37^\circ \ \text{ with } (-\text{ve})\, x-\text{axis}$$

Problem # 2

Determine the magnitude and direction of the anchoring force needed to hold the horizontal elbow and nozzle combination shown in place as shown in figure. Atmospheric pressure is 100 kPa (abs). The gage pressure at section (1) is 100 kPa. At section (2), the water exits to the atmosphere.



Solution: from steady flow momentum principle;

$$\boxed{\Rightarrow \sum \vec{F}_{CV} = \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\rightarrow x: \sum F_x = \dot{m}\left(V_{2x} - V_{1x}\right)$$

$$\Rightarrow -R_x + p_1 A_1 = \dot{m}\left(V_{2x} - V_{1x}\right) = \rho V_1 A_1\left(-V_2 - V_1\right) \quad ; \qquad \left(V_2 \leftarrow \; -ve\right)$$

$$\Rightarrow R_x = p_1 A_1 + \left(\rho V_1 A_1\right)\left(V_1 + V_2\right)$$

Now, from continuity equation: $Q_1 = Q_2 \quad \Rightarrow A_1 V_1 = A_2 V_2$

$$\Rightarrow \left(\frac{\pi}{4} d_1^2\right)(V_1) = \left(\frac{\pi}{4} d_2^2\right)(V_2)$$

$$\Rightarrow V_2 = \left(\frac{d_1}{d_2}\right)^2 V_1$$

$$\therefore V_2 = 7.03 \text{ m/s}$$

Problem # 2 cont…

Then,

$$\rightarrow x: \ \Rightarrow R_x = p_1 A_1 + \left(\rho V_1 A_1\right)\left(V_1 + V_2\right)$$

$$\Rightarrow F_x = (100\times10^3)\left(\frac{\pi}{4}\times0.3^2\right) + \left(1000\times2\times\frac{\pi}{4}\times0.3^2\right)(2+7.03)$$

$$\therefore \ F_x = 8.35\,\text{kN}$$

Now, along y-axis:

$$\uparrow y: \ -R_y = \dot{m}\left(V_{2y} - V_{1y}\right) = \dot{m}\left(0-0\right) = 0 \quad \therefore R_y = 0$$

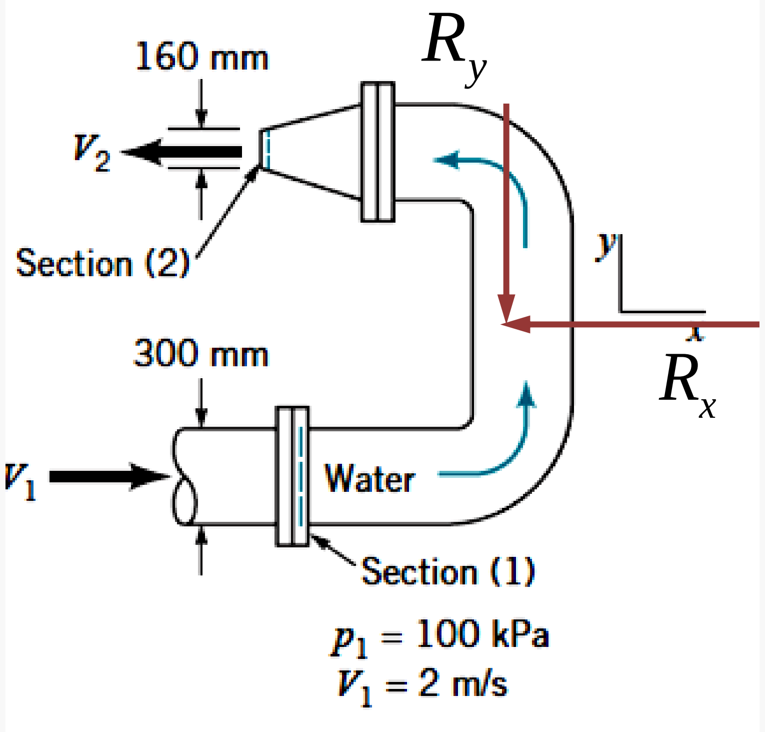
So, the magnitude of resultant anchoring force is

$$R = \sqrt{R_x^2 + R_y^2}$$

$$\Rightarrow R = \sqrt{8.35^2 + 0^2}$$

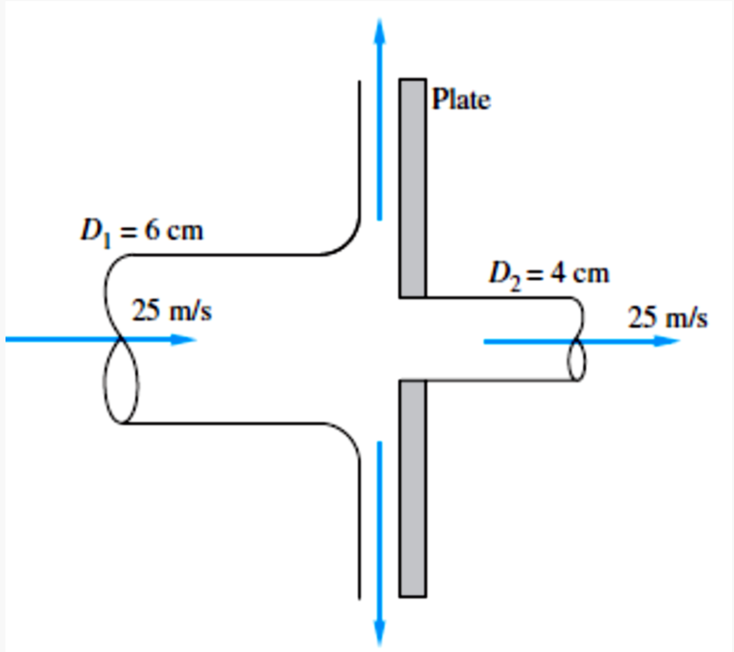
$$\Rightarrow R = 8.35\,\text{kN}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \ 0^\circ \ \ \text{with}\left(-\text{ve}\right) x - \text{axis}$$



Problem # 3

The 6-cm-diameter 20°C water jet strikes a plate containing a hole of 4-cm diameter. Part of the jet passes through the hole, and part is deflected. Determine the horizontal force required to hold the plate.



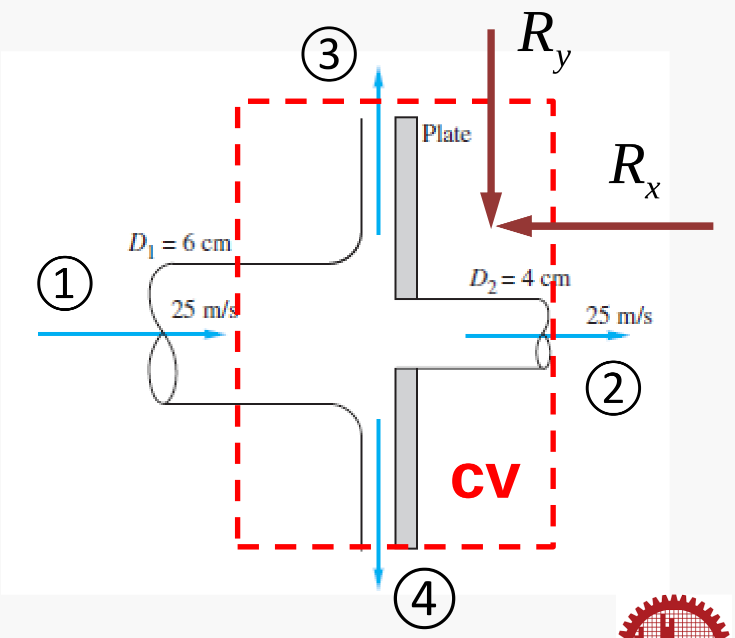
Solution:

$$Q_1 = \frac{\pi}{4} d_1^2 V_1 = \frac{\pi}{4}(0.06)^2(25) = 0.0707 \ \text{m}^3/\text{s}$$

$$Q_2 = \frac{\pi}{4} d_2^2 V_2 = \frac{\pi}{4}(0.04)^2(25) = 0.0314 \ \text{m}^3/\text{s}$$

from steady flow momentum principle:

$$\boxed{\sum \vec{F}_{\text{CV}} = \int_{\text{CS}} \vec{\mathbf{V}} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}} \right) dA}$$



$$\rightarrow x: \sum F_x = \left(\dot{m} V_x\right)_{out} - \left(\dot{m} V_x\right)_{in}$$

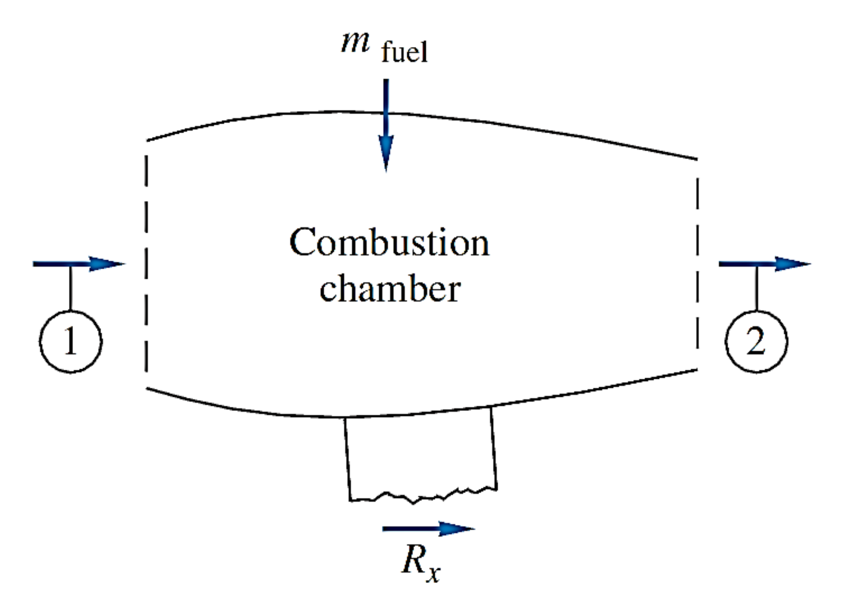
$$\Rightarrow -R_x = \left(\dot{m}_2 V_{2x} + \dot{m}_3 V_{3x} + \dot{m}_4 V_{4x}\right) - \dot{m}_1 V_{1x}$$

$$\Rightarrow -R_x = \left((1000)(0.0314)(25) + \dot{m}_3(0) + \dot{m}_4(0)\right) - (1000)(0.0707)(25)$$

$$\therefore \quad R_x = 982.5\,\text{N} \qquad \text{(to left)} \qquad \text{Ans.}$$

Problem # 4

(a) The jet engine on a test stand admits air at 20°C and 1 atm at section 1, where A_1 = 0.5 m^2 and V_1 = 250 m/s. The fuel-to-air ratio is 1:30. The air leaves section 2 at atmospheric pressure and higher temperature, where V_2 = 900 m/s and A_2 = 0.4 m^2. Compute the horizontal test stand reaction $\mathbf{R_x}$ needed to hold this engine fixed.



Solution:

$$\rho_1 = \frac{p}{RT} = \frac{101325}{(287)(273+20)} = 1.205 \text{ kg/m}^3$$

$$\dot{m}_1 = \rho_1 A_1 V_1 = (1.205)(0.5)(250) \;=\; 150.6 \text{ kg/s}$$

The fuel-to-air ratio is 1:30;

$$\therefore\ \dot{m}_2 = 150.6\left(1+\frac{1}{30}\right) = \; 155.6 \text{ kg/s}$$

Problem # 4

from steady flow momentum principle:

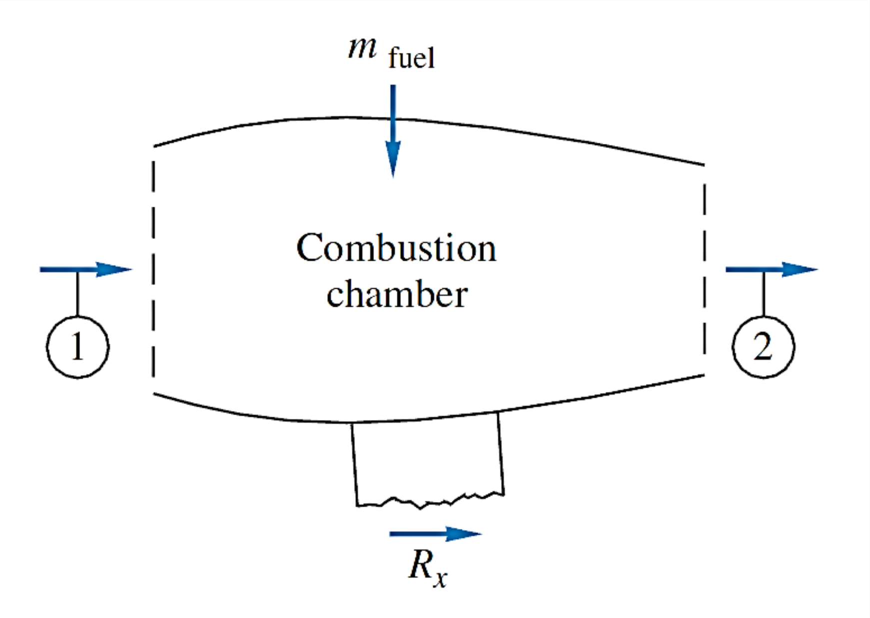
$$\boxed{\sum \vec{F}_{\text{CV}} = \int_{\text{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\rightarrow x: \sum F_x = \left(\dot{m}V_x\right)_{out} - \left(\dot{m}V_x\right)_{in}$$

$$\Rightarrow R_x = \dot{m}_2 V_{2x} - \left(\dot{m}_1 V_{1x} + \dot{m}_{fuel} V_{f,x}\right)$$

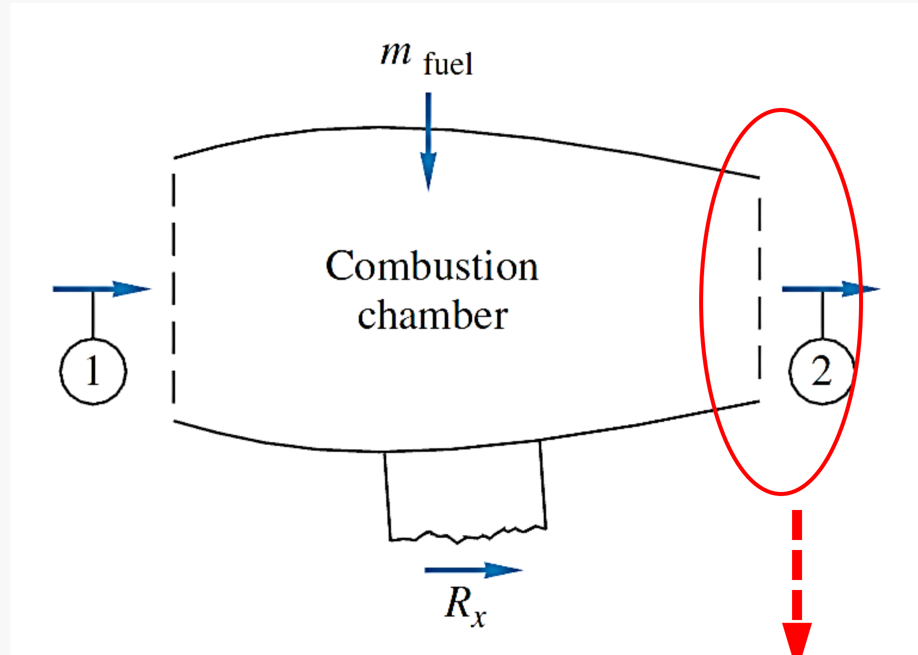
$$\Rightarrow R_x = (155.6)(900) - \left((150.6)(250) + \dot{m}_{fuel}(0)\right)$$

$$\therefore \quad R_x = 102.4 \text{ kN} \qquad \text{(to right)} \qquad \text{Ans.}$$



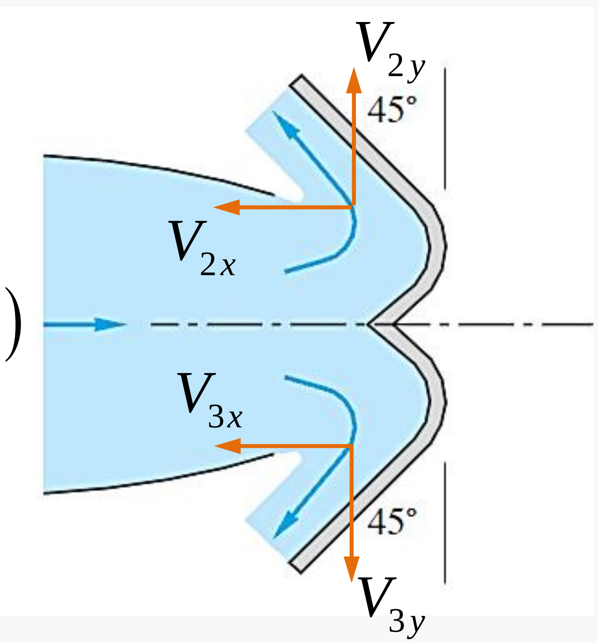
Problem # 4

(b) Suppose that a deflector is deployed at the exit of the jet engine of Prob. 5(a), as shown in Figure. What will the reaction $\mathbf{R_x}$ on the test stand be now?



from steady flow momentum principle:

$$\boxed{\sum \vec{F}_{\mathrm{CV}} = \int_{\mathrm{CS}} \vec{\mathbf{V}} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}} \right) dA}$$



$$\rightarrow x : \sum F_x = \left(\dot{m} V_x\right)_{out} - \left(\dot{m} V_x\right)_{in}$$

$$\Rightarrow R_x = \left(\dot{m}_2 V_{2x} + \dot{m}_3 V_{3x}\right) - \left(\dot{m}_1 V_{1x} + \dot{m}_{fuel} V_{f,x}\right)$$

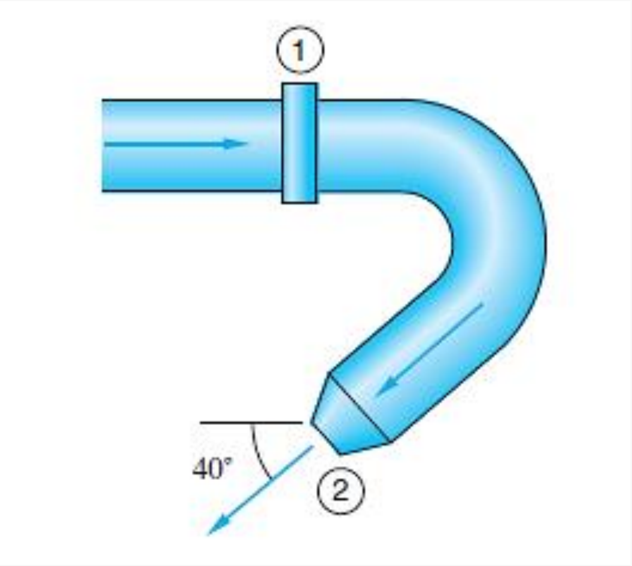
$$\Rightarrow R_x = \left(\left(\frac{155.6}{2}\right)(-900\cos 45°) + \left(\frac{155.6}{2}\right)(-900\cos 45°) \right) - \left((150.6)(250) + \dot{m}_{fuel}(0)\right)$$

$$\Rightarrow R_x = -137.6 \text{ kN}$$

$$\therefore \quad R_x = 137.6 \text{ kN} \ \ \text{(to left)} \qquad \text{Ans. (b)}$$

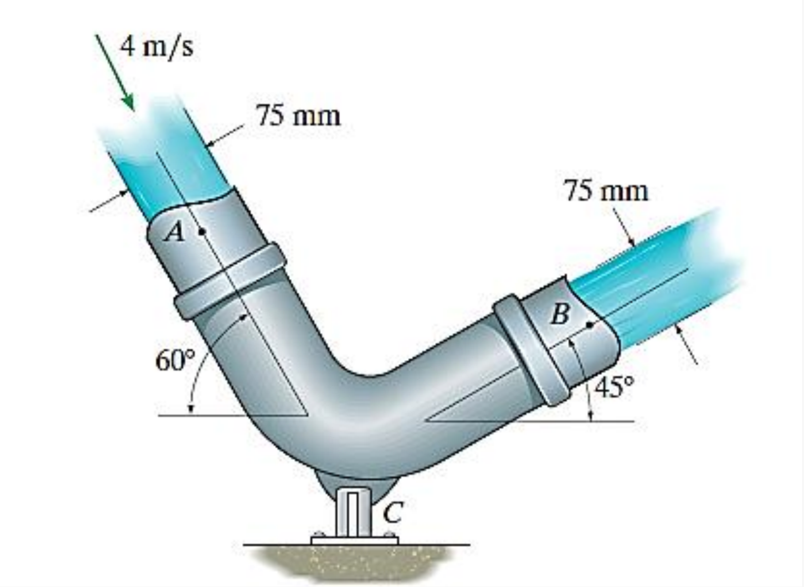
Problem # 5

Water at 20°C flows through the elbow as shown in Fig. and exits to the atmosphere. The pipe diameter is D_1 = 10 cm, while D_2 = 3 cm. At a weight flow rate of 150 N/s, the pressure p_1 = 2.3 atm (gage). Neglecting the weight of water and elbow, estimate the force on the flange bolts at section 1.



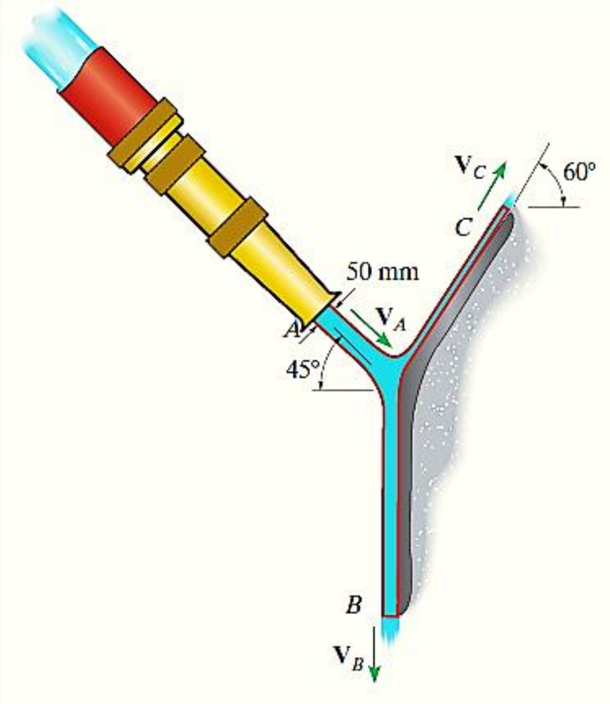
Problem # 6

Water flows through the elbow with a velocity of 4 m/s. Determine the horizontal and vertical components of force the support at *C* exerts on the elbow. The pressure within the pipe at *A* and *B* is 200 kPa. The flow occurs in the horizontal plane. Assume there is no support at *A* and *B*.

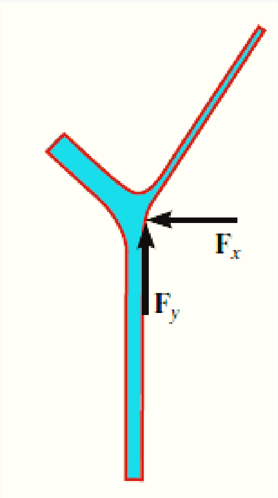


Problem # 7

Water is discharged from the 50-mm-diameter nozzle of a fire hose at 15 liters/s as shown in Fig. The flow strikes the fixed surface such that 3/4 flows along *B*, while the remaining 1/4 flows along *C*. Determine the *x* and *y* components of the resultant force exerted on the surface. Assume steady flow.



Hints:



Neglect elevation difference and existing free surface at point A, B and C gives,
$V_A = V_B = V_C$

Ans:

$$F_x = 66.70\ \text{N} = 66.7\ \text{N} \leftarrow$$

$$F_y = 19.89\ \text{N} = 19.9\ \text{N} \uparrow$$